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Regular and chaotic motion of fluid particles in a two-dimensional fluid

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Abstract. We study the chaotic behaviour exhibited by particles which move in a two-dimensional fluid. The connection of this Lagrangian chaos with the velocity field behaviour is discussed both in the Lorenz model and in truncated Navier-Stokes equations. We indicate a possible method for the onset of Lagrangian chaos which seems to be rather generic. Lagrangian chaos appears when the Eulerian equation passes from a steady solution to a periodic one via Hopf bifurcation. It is also shown that the transition to chaos for the velocity field ('Eulerian chaos') does not affect the particle motion properties in some typical cases.

1. Introduction

In the Lagrangian description of the dynamics of a fluid one deals with the trajectory, say $\boldsymbol{x}^{(i)}(t)$, of each fluid particle i of the fluid. This approach, and the Eulerian one in which one considers the time evolution of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, are of course related by the differential equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}, t) \tag{1}$$

and are thus equivalent, at least in principle. Let us stress that this does not mean that, if the velocity field exhibits a regular behaviour, the motion of a particle in the fluid should not be chaotic. Indeed, in three dimensions there are cases for which the particle motion can be chaotic, even if the velocity field is stationary [1, 2], i.e. $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$. It should not be too surprising, since equation (1) is a non-linear dynamical system where deterministic chaos can arise.

We note that (1) also describes the motion of a powder particle immersed in the fluid if this particle is small enough not to disturb the velocity field but also big enough not to perform a Brownian motion.

The understanding of the Lagrangian chaos and of (possible) relations with the corresponding Eulerian behaviour is still at a rough level, although this problem is relevant both on the theoretical and applicative sides, e.g. in diffusion problems [3].

At present, there are only a few works which mainly investigate stationary velocity fields (ABC flows) in three dimensions [1, 2] and velocity fields which are periodic in time in two dimensions [4].

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We have thus decided to study two-dimensional Lagrangian behaviours in different Eulerian regimes. In particular, our analysis is focused both on the onset of the Lagrangian chaos and on the consequence of the onset of turbulence (in the velocity field) on the Lagrangian behaviour. We have considered the evolution of incompressible viscous fluids, by taking into account truncations of the Navier-Stokes equations:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f} \tag{2}$$

where ρ is the density, p is the pressure, ν the kinematic viscosity and f an external forcing. As we consider truncations with few modes, we can have only temporal chaos of the velocity field; indeed the spatial structure is highly coherent.

The truncated models have been exhaustively studied by varying the Reynolds number *Re* by Franceschini and coworkers [5-9] and we recall their derivation in appendix 1. Our dynamical system is thus given by $2+F$ equations:

$$\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t} = \boldsymbol{f}_{Re}(\boldsymbol{\gamma}) \qquad \text{with } \boldsymbol{\gamma}, \boldsymbol{f} \in R^F \tag{3a}$$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\gamma}(t)) \qquad \text{with } \boldsymbol{x} \in R^2 \tag{3b}$$

where $\boldsymbol{\gamma}$ are the variables which give the velocity field corresponding to the F modes considered, one of which is excited by an external forcing.

Let us stress that in two dimensions the incompressibility condition is satisfied by assuming

$$u_1 = \frac{\partial \psi}{\partial x_2} \qquad u_2 = -\frac{\partial \psi}{\partial x_1} \tag{4}$$

where $\psi = \psi(\boldsymbol{x}, t)$ is the stream function. ψ is thus formally the time-dependent Hamiltonian of the dynamical system given by equation (3*b*).

We measure the degree of chaos of the system by the computation of three different Lyapunov exponents: λ_{E} for the Eulerian part; λ_{L} for the Lagrangian part, assuming that the evolution $\boldsymbol{\gamma}(t)$ is known; λ_{T} of the full system given by $2+F$ evolution equations. The Lyapunov exponents can be defined [10] by the relation:

$$\lambda_{\mathrm{E,L,T}} = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|z(t)^{\mathrm{E,L,T}}|}{|z(0)^{\mathrm{E,L,T}}|} \tag{5}$$

where the evolution of the tangent vectors z are given by the linearised equations

$$\frac{\mathrm{d}z_i^{(\mathrm{E})}}{\mathrm{d}t} = \sum_{j=1}^{F} \left.\frac{\partial f_i}{\partial \gamma_j}\right|_{\gamma(t)} z_j^{(\mathrm{E})} \qquad z^{(\mathrm{E})} \in R^F \tag{6a}$$

for the Eulerian part,

$$\frac{\mathrm{d}z_i^{(\mathrm{L})}}{\mathrm{d}t} = \sum_{j=1}^{2} \left.\frac{\partial u_i(\boldsymbol{x}, \boldsymbol{\gamma}(t))}{\partial x_j}\right|_{\boldsymbol{x}(t)} z_j^{(\mathrm{L})} \qquad z^{(\mathrm{L})} \in R^2 \tag{6b}$$

for the Lagrangian part, and

$$\frac{\mathrm{d}z_i^{(\mathrm{T})}}{\mathrm{d}t} = \sum_{j=1}^{F+2} \left.\frac{\partial G_i}{\partial y_j}\right|_{y(t)} z_j^{(\mathrm{T})} \qquad z^{(\mathrm{T})} \in R^{(F+2)} \tag{6c}$$

for the whole system (2) with $y = (\gamma_1, \ldots, \gamma_F, x_1, x_2)$ and $G = (f_1, \ldots, f_F, u_1, u_2)$. The meaning of these Lyapunov exponents is quite clear: λ_E estimates the exponential rate of the imprecision increasing in the velocity field knowledge, while the distance between two particles in a fluid increases as $\exp(\lambda_L t)$, when the Eulerian part is given (see appendix 2 for a discussion of this point). On the other hand, if the velocity field is not known the uncertainty of the position of a particle increases with an exponential rate given by λ_T.

In general there is no relation between λ_E and λ_L while simple arguments show that

$$\lambda_T = \max(\lambda_E, \lambda_L). \tag{7}$$

Let us in fact note that $z^{(T)} = (z^{(E)}, \zeta^{(L)})$ where the evolution equation of $\zeta^{(L)} \in R^2$ can be written as

$$\frac{d\zeta_i^{(L)}}{dt} = \sum_{j=1}^{2} \frac{\partial u_i}{\partial x_j} \zeta_j^{(L)} + \sum_{k=1}^{F} \frac{\partial u_i}{\partial \gamma_k} z_k^{(E)}. \tag{8}$$

Since $|z^{(E)}| \propto \exp(\lambda_E t)$ one has that the second summation in (8) is $O(e^{\lambda_E t})$. It follows that if $\lambda_L > \lambda_E$ the second summation in (8) is negligible with respect to the first one and $\lambda_T = \lambda_L$. In the opposite case $\lambda_E > \lambda_L$, equation (8) has the same form of (6*b*) with a forcing term $O(e^{\lambda_E t})$ which is much larger than the 'autonomous' term

$$\sum^{2} \frac{\partial u_i}{\partial x_j} \zeta_j^{(L)}$$

and so $\lambda_T = \lambda_E$. In the truncated models of the Navier–Stokes equations we have always found that $\lambda_L > \lambda_E$, which seems to be a rather generic situation.

In § 2 we nevertheless discuss an atypical example contrasting with the above standard case. We show that the Eulerian chaoticity of the Lorenz model [11] does not imply a chaotic motion of particles, i.e. one has $\lambda_E > 0$ while $\lambda_L = 0$. This is, however, due to the very particular form of the velocity field but it is worth stressing that such a non-generic feature arises in one of the most typical examples of chaotic systems with few degrees of freedom.

In § 3, we discuss the case of more generic Lagrangian behaviours obtained by varying the Reynolds number in non-trivial truncations of the Navier–Stokes equations. In § 4 the reader will find some concluding remarks. We study the features of the particle motion around:

(i) the transition of the Eulerian flow from the stable stationary state to the periodic motion by a Hopf bifurcation;

(ii) the transition of the Eulerian flow from a quasiperiodic state to a chaotic motion.

In the first case the stream function passes from a time-independent form to a periodic one. Note that equation (3*b*) describes a two-dimensional Hamiltonian system and one expects to observe the typical onset of chaos of these systems.

In case (ii) the transition to chaos (in the Eulerian sense) seems to have no effect on the Lagrangian behaviour, i.e. λ_L does not change with the Reynolds number value. This result is important from an experimental point of view, as it means that measurements of particle positions in a fluid do not allow us to determine the presence of chaotic attractors for the Eulerian part.

In appendix 1 we sketch the derivation of the truncated models and in appendix 2 we comment on the problem of the divergence of particles which are initially close since there are some controversial positions on this in the literature [12–14].

2. The Lorenz model: an atypical example

We want to discuss here the Lorenz model behaviour which appears to be quite surprising. In fact, it does not exhibit Lagrangian chaos even when it is chaotic in the Eulerian sense. This is due to the particular form of the evolution equations for the velocity field and contrasts with generic situations. Nevertheless, the Lorenz model is, for historical reasons, one of the most widely studied chaotic dynamical systems. It is thus interesting to point out the peculiar Lagrangian motion originated by it.

Let us recall that the Lorenz model is obtained by a simplification of the equations which rule the atmospheric convection. The gravitational field and the temperature difference are assumed to be directed along x_2 with $0 \leq x_2 \leq \pi$ and there is a periodicity in direction 1. One considers three degrees of freedom ($F = 3$), i.e. three variables γ_i related to the temperature gradient and to the stream function as follows:

$$\psi = \gamma_1 \sin(x_1) \sin(x_2) \tag{9a}$$

$$\delta T = 2^{1/2} \gamma_2 \cos(x_1) \sin(x_2) - \gamma_3 \sin(2x_2) \tag{9b}$$

where δT is the departure of the temperature from the linear behaviour in the non-convective case.

Equations (3) can thus be written as

$$\frac{d\gamma_1}{dt} = -\sigma\gamma_1 + \sigma\gamma_2 \tag{10a}$$

$$\frac{d\gamma_2}{dt} = R\gamma_1 - \gamma_2 - \gamma_1\gamma_3 \tag{10b}$$

$$\frac{d\gamma_3}{dt} = \gamma_1\gamma_2 - \tfrac{8}{3}\gamma_3 \tag{10c}$$

$$\frac{dx_1}{dt} = u_1 = \partial_2\psi = 2^{1/2}\gamma_1(t) \sin(x_1) \cos(x_2) \tag{10d}$$

$$\frac{dx_2}{dt} = u_2 = -\partial_1\psi = -2^{1/2}\gamma_1(t) \cos(x_1) \sin(x_2). \tag{10e}$$

Here R is the Rayleigh number which plays a role analogous to that of the Reynolds number in the Navier–Stokes equations, σ is the Prandt number and x_i, t and γ_i are rescaled non-dimensional variables. Since the velocity field has zero perpendicular component on the boundaries of the square $[0, \pi] \times [0, \pi]$ and it is periodic in x_1, the particle is confined in $[\pi n, \pi(n+1)] \times [0, \pi]$ for each n value, if it is in this square at the initial time. We can thus limit ourselves to study the zone $[0, \pi] \times [0, \pi]$.

One can show that $\lambda_L = 0 \, \forall R$ and that the orbit (x_1, x_2) is always closed and depends only on the initial condition and not on R; see figure 1. We can easily explain this feature of the orbits, for example by considering an initial condition near the Lagrangian fixed point $(\pi/2, \pi/2)$, say $(\pi/2 + q_1, \pi/2 + q_2)$, with q_i small enough. We thus get from (10*d*, *e*):

$$\frac{dq_1}{dt} = -2^{1/2}\gamma_1(t) q_2 \tag{11a}$$

$$\frac{dq_2}{dt} = 2^{1/2}\gamma_1(t) q_1. \tag{11b}$$

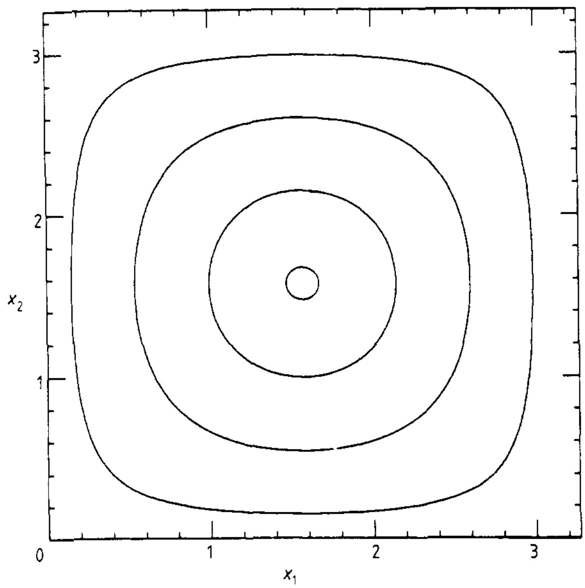

Figure 1. Structure of Lagrangian orbits (x_1, x_2) for the Lorenz model (10).

By integrating (11) using polar coordinates we see that $r(t) = \text{constant}$ while the phase is $\phi(t) = 2^{1/2} \int \gamma_1(\tau)\, d\tau$. The particle therefore moves along a circle of radius r but with an angular velocity $2^{1/2}\gamma_1(t)$ which is chaotic for some R values.

It follows that $\lambda_L = 0$ since two particles which are initially close do not diverge as both δr and $\delta\phi(t)$ are constant. The previous result, obtained near the Lagrangian fixed point, holds for each initial condition because one has from (10*d*, *e*) that the derivative

$$\frac{dx_2}{dx_1} = g(x_1, x_2) \tag{12}$$

depends neither on R nor on t. Therefore one has that the orbit structure of (10*d*, *e*) coincides with that of an autonomous system with $\gamma_1 = \text{constant}$. The Bendixson–Poincaré theorem [15] and the incompressibility constraint div $u = 0$ then ensure that the form of the orbits has to be closed. Moreover one can verify that (10*d*, *e*) have the integral of motion:

$$h(\boldsymbol{x}) = \sin(x_1)\sin(x_2) = \text{constant}. \tag{13}$$

Let us consider two particles, initially closed, in the same velocity field so that $h(\boldsymbol{x}^{(1)}) \simeq h(\boldsymbol{x}^{(2)})$. Moreover, the two particles describe closed orbits. Therefore in order to have an exponential growth of the distance one has to request that the 'phase' difference $\delta\phi = (\phi^{(1)} - \phi^{(2)})$ increases exponentially. One can see that this is not possible. In fact $d\phi/dt = \Omega(h, \gamma_1)$ and since γ_1 is the same for the two particles one has $|\delta\phi(t)| < \text{constant} \times t + |\delta\phi(0)|$. Therefore one obtains $\lambda_L = 0$ even if $\lambda_E > 0$. This means that the imprecision in the knowledge of the particle position does not increase exponentially once the velocity field is specified. Two particles therefore remain close in the same Eulerian field realisation. Nevertheless the particle motion appears irregular if $\lambda_E > 0$, as shown in figure 2. Let us in fact stress that in general we do not know with infinite precision the velocity field (i.e. the $\boldsymbol{\gamma}$) and this could involve an exponential

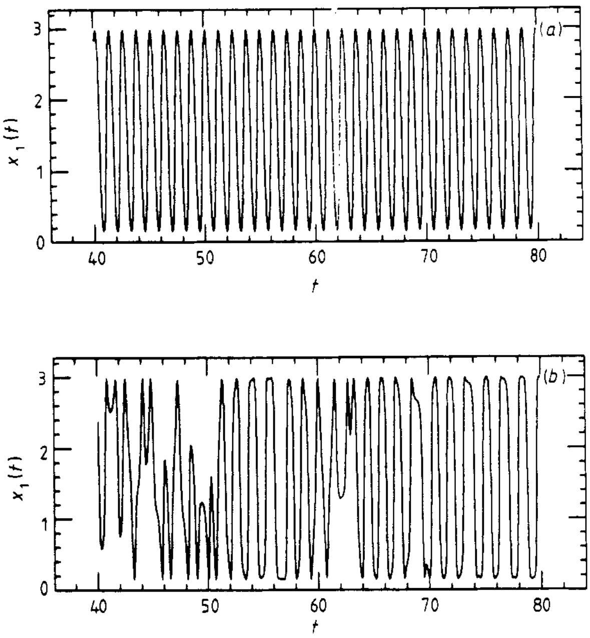

Figure 2. $x_1(t)$ against t for the Lorenz model: (a) $R=22$, $\sigma=10$ (the Eulerian part ($10a, b, c$) is regular); (b) $R=26.24$, $\sigma=10$ (the Eulerian part is chaotic).

instability in $\boldsymbol{x}(t)$; e.g. in the case with the initial condition near the Lagrangian fixed point one has

$$\delta\phi(t)=2^{1/2}\int\delta\gamma_1(\tau)\,\mathrm{d}\tau\propto\exp(\lambda_E t)$$

where $\delta\phi$ is now the difference between the phase of the particle in the velocity field given by $\boldsymbol{\gamma}$ and that in the velocity field given by $\boldsymbol{\gamma}+\delta\boldsymbol{\gamma}$.

We, however, expect that the opposite situation $\lambda_E=0$ and $\lambda_L>0$ is the typical one for low values of the control parameter (e.g. the Reynolds number in the Navier-Stokes equations) as it corresponds to the presence of chaotic particle motions in regular velocity fields.

3. Irregular motions in two-dimensional fluids

This section examines the motion of particles in a two-dimensional incompressible fluid described by the Navier-Stokes equations. We shall limit ourselves to the study of the so-called truncated models. Let us consider periodic boundary conditions and develop the stream function ψ in a Fourier series taking into account only F modes (see appendix 1 for the details):

$$\psi=-\mathrm{i}\sum_j|\boldsymbol{k}_j|^{-1}\gamma_{k_j}\exp(\mathrm{i}\boldsymbol{k}_j\cdot\boldsymbol{x})+\mathrm{cc}. \tag{14}$$

By inserting the velocity field obtained by (14) into the Navier-Stokes equations we have a system of non-linear differential equations of the form ($3a$):

$$\frac{\mathrm{d}\gamma_i}{\mathrm{d}t}=-k_i^2\gamma_i+\sum_{l,m}A_{ilm}\gamma_l\gamma_m+f_i \tag{15}$$

with $i = 1, 2, \ldots F$. We have studied the Lagrangian behaviour (3*b*) exhibited by this system for $F = 5$ and $F = 7$, using the results previously obtained for the Eulerian part.

In the case $F = 5$ for $Re < Re_1 = 22.853\,701\,63\ldots$, there are four stable stationary solutions $\tilde{\boldsymbol{\gamma}}$. At $Re = Re_1$ these solutions become unstable, four stable periodic orbits arise via Hopf bifurcation [16], and one finds that:

$$\boldsymbol{\gamma}(t) = \tilde{\boldsymbol{\gamma}} + (Re - Re_1)^{1/2}\delta\boldsymbol{\gamma}(t) + \mathrm{O}(Re - Re_1) \tag{16}$$

where $\delta\boldsymbol{\gamma}(t)$ is periodic with period $T(Re) = T_0 + \mathrm{O}(Re - Re_1)$ and $T_0 = 0.732\,27\ldots$. These limit cycles are stable up to $Re_2 = 28.41\ldots$, where they bifurcate to new double period orbits. A cascade of bifurcations takes place giving rise to new multiperiod orbits up to a critical value $Re_c \approx 28.73$ above which chaotic attractors appear and λ_E becomes larger than zero.

The stream function for $Re < Re_1$ is asymptotically stationary, i.e. $\psi(\boldsymbol{x}, t) \to \psi(\boldsymbol{x})$. It follows from the Bendixson-Poincaré theorem [15] and from the divergenceless condition $\operatorname{div} \boldsymbol{u} = 0$ that the solutions $\boldsymbol{x}(t)$ of the equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{\nabla}^{\perp}\psi \qquad \boldsymbol{\nabla}^{\perp} = (\partial_2, -\partial_1)$$

have a regular behaviour with fixed points, or closed orbits, or going to infinity orbits. Figure 3 shows the orbit structure for $Re = Re_1 - 0.05$ and

$$\tilde{\boldsymbol{\gamma}} = \{-(\tfrac{5}{3})^{1/2}, -\tfrac{3}{80}(\tfrac{5}{3})^{1/2}Re_1, \tfrac{9}{80}Re_1, \tfrac{1}{3}[(\tfrac{9}{80}Re_1)^2 - \tfrac{3}{2}]^{1/2}, (\tfrac{5}{3})^{1/2}[(\tfrac{9}{80}Re_1)^2 - \tfrac{3}{2}]^{1/2}\}.$$

The structure of the separatrices and of the hyperbolic fixed points is well evident. There are two kinds of separatrices: the isolated 'eights' labelled by A and the periodic ones labelled by B in figure 3.

For $Re = Re_1 + \varepsilon$ the stream function becomes time dependent with the form

$$\psi(\boldsymbol{x}, t) = \tilde{\psi}(\boldsymbol{x}) + \varepsilon^{1/2}\delta\psi(\boldsymbol{x}, t) + \mathrm{O}(\varepsilon) \tag{17}$$

where $\tilde{\psi}$ is given by $\tilde{\boldsymbol{\gamma}}$ and $\delta\psi$ by $\delta\boldsymbol{\gamma}$ and is thus periodic with period T. It is well known that in one-dimensional time-dependent Hamiltonian systems, the onset of

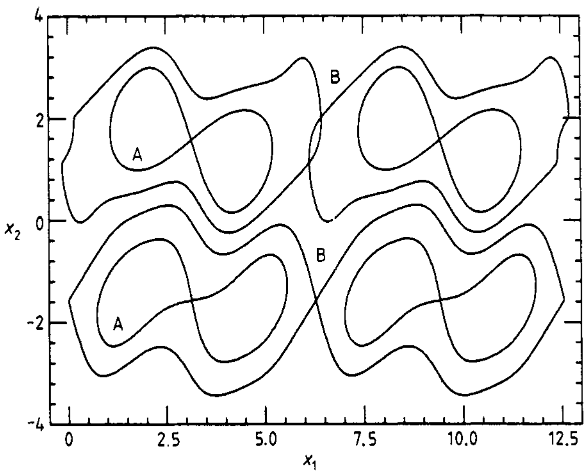

Figure 3. Structure of the separatrices for equation (3*b*) with ψ given by the model (see equation (A1.3)).

chaos is typically observed to begin around the separatrix by unfolding and crossing of the stable and unstable manifolds [17]. There exists a method due to Melnikov [18] in order to rigorously prove that the motion is chaotic in a small region around separatrices, but it requires the explicit knowledge of the unperturbed trajectory on the separatrices. In our case this is not trivial, because of the complicated structure of the Hamiltonian, i.e. ψ.

Figure 4 shows that for small ε the Lyapunov exponent λ_L is positive only in a tiny region around the separatrices while it vanishes in the other regions. One can pictorially express (see figure 5) the chaotic and regular motions by means of a Poincaré section given by

$$x(nT) \to x(nT+T) \tag{18}$$

and we numerically compute the period $T(\varepsilon)$. The size of the chaotic region around

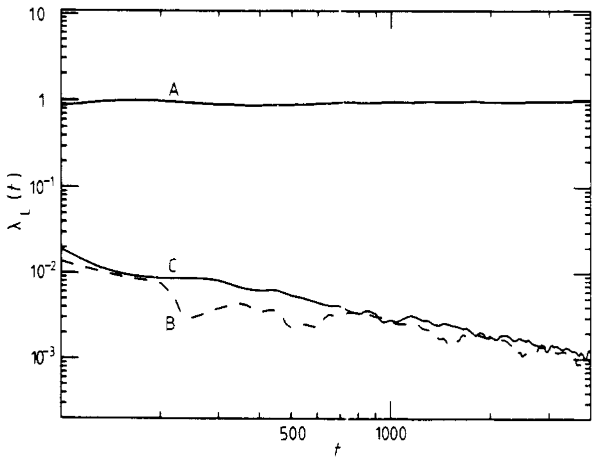

Figure 4. $\lambda_L(t) = (1/t)\ln(|z^L(t)|/|z^L(0)|)$ against t at $\varepsilon = 0.05$ with different conditions: A, $(x_1(0), x_2(0)) = (3.2, -1.6)$, close to a separatrix; B, $(x_1(0), x_2(0)) = (4.3, -2.0)$, far from the separatrices; C, $(x_1(0), x_2(0)) = (4.267, -3.009)$, far from the separatrices.

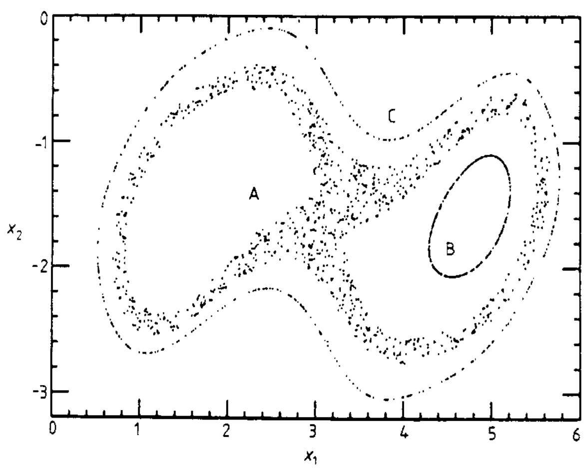

Figure 5. Poincaré section (18) of the trajectories with the initial conditions of figure 4.

the separatrices increases with ε and at large values of $\varepsilon(\approx 0.7)$ it is practically impossible to distinguish between a regular region and a chaotic one.

For small ε there are three types of behaviour with respect to diffusion.

(i) A motion confined inside the separatrices of kind B, where the motion is regular or chaotic according to the distance from the 'eights'.

(ii) A regular motion that goes to infinity, in the region out and far enough from the separatrices of kind B.

(iii) A chaotic motion near these separatrices with non-trivial diffusive features.

At large ε values all the regions become connected and one observes just one diffusive behaviour.

We want to stress that, while the diffusion behaviours are strongly related to the detailed structures of the velocity fields, this scenario for the onset of the Lagrangian chaos should be quite generic. Indeed the features which we have described are strictly related to the separatrix structure and to the mechanism for the appearance of chaos in one-dimensional time-dependent Hamiltonian systems. We therefore expect in 2D this kind of behaviour whenever Hopf bifurcations are exhibited by the Eulerian part. Note that this is just the case for the whole class of truncated models of the Navier-Stokes equations.

However, the transition to the Eulerian chaoticity depends on the particular model considered, e.g. in the five-modes truncated model it happens via period doubling and moreover the system again becomes regular for a further increase of the Reynolds number. The seven-modes model has a different kind of transition (i.e. collapsing of periodic orbits) and there the system does not come back to a regular velocity field at larger *Re*.

We have therefore computed λ_E and λ_L as a function of *Re* in both these models. In figure 6 it is well evident that λ_L is not affected at all by the sharp transition to Eulerian chaoticity in the five-modes model. The same qualitative behaviour has been observed for the seven-modes model around the critical value $Re_c \approx 555$.

This suggests that the onset of the Eulerian chaos has no influence on the Lagrangian properties also in two-dimensional fluids.

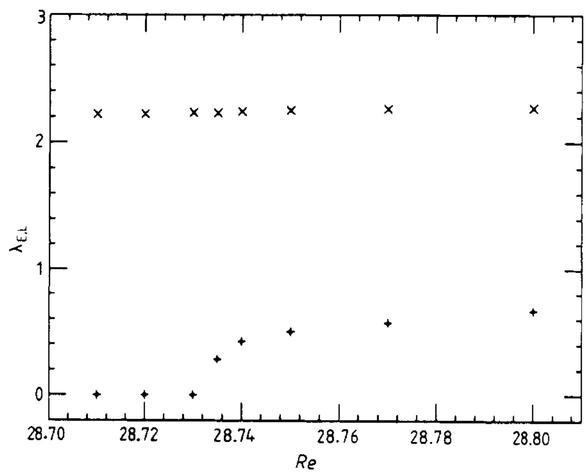

Figure 6. λ_E, λ_L against *Re* for the five-modes model around Re_c: ✦, Eulerian LE; ×, Lagrangian LE.

We want to note that the above scenario does not hold in 3D; we recall also that the case with a stationary velocity field can give a Lagrangian chaotic behaviour [1, 2]. Therefore we do not expect a generic mechanism for the transition to Lagrangian chaos in 3D similar to one due to the Eulerian Hopf bifurcation, except perhaps in some very particular situations [19].

4. Conclusions

We have seen that in two-dimensional incompressible fluids the Lagrangian behaviour has a clear connection with the structure of the velocity field only when the Eulerian part exhibits a Hopf bifurcation. In the other cases no simple relations can be found. We have in particular shown that a system can be chaotic in the Lagrangian sense without a chaotic velocity field (which is the typical case) but it is also possible in the opposite (peculiar) situations; see the discussion on the Lorenz model.

Let us stress that one cannot separate the Lagrangian and Eulerian properties by experimental measurements involving only the motion of one particle (e.g. a buoy in oceanic currents [20]). Indeed, using standard signal treatment methods [21] one can extract the total Lyapunov exponent λ_{T} but not λ_{L} or λ_{E}. However, in generic cases one expects that $\lambda_{\mathrm{T}}=\lambda_{\mathrm{L}}$, but there exist counterexamples as we have discussed in § 2.

Acknowledgment

It is a pleasure to thank our friend Andrea Crisanti.

Appendix 1

We briefly discuss a quite standard procedure for obtaining a finite degrees of freedom approximation of the Navier-Stokes equations (2) [9]. Let us consider periodic boundary conditions on a square of edge 2π and expand $\boldsymbol{u}$ in the Fourier series taking into account the incompressibility condition:

$$\boldsymbol{u}(\boldsymbol{x})=\sum_{\boldsymbol{k}} \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x})]\gamma_{\boldsymbol{k}}\frac{\boldsymbol{k}^{\perp}}{|\boldsymbol{k}|} \tag{A1.1}$$

where $\boldsymbol{k}=(k_1,k_2)$, $\boldsymbol{k}^{\perp}=(k_2,-k_1)$ and $\gamma_{\boldsymbol{k}}=-\gamma^{*}_{-\boldsymbol{k}}$ because of the reality of $\boldsymbol{u}(\boldsymbol{x})$. By expanding p and f in a similar way one has the evolution equations for a truncation L of $\gamma_{\boldsymbol{k}}$

$$\frac{\mathrm{d}\gamma_{\boldsymbol{k}}}{\mathrm{d}t}=-\mathrm{i}\sum_{\substack{\boldsymbol{k}'+\boldsymbol{k}''+\boldsymbol{k}=0\\(\boldsymbol{k}',\boldsymbol{k}'',\boldsymbol{k}\in L)}}\frac{(\boldsymbol{k}')^{\perp}\cdot(\boldsymbol{k}'')(|\boldsymbol{k}''|^2-|\boldsymbol{k}'|^2)}{2|\boldsymbol{k}'||\boldsymbol{k}''||\boldsymbol{k}|}\gamma^{*}_{\boldsymbol{k}'}\gamma^{*}_{\boldsymbol{k}''}-\nu|\boldsymbol{k}|^2\gamma_{\boldsymbol{k}}+f_{\boldsymbol{k}} \tag{A1.2}$$

where L is a set of wavevectors $\boldsymbol{k}$ such that if $\boldsymbol{k}\in L$ then $-\boldsymbol{k}\in L$. Let us define the modes $1,2,\ldots,9$ corresponding to $\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_9$ with $\boldsymbol{k}_1=(1,1)$, $\boldsymbol{k}_2=(3,0)$, $\boldsymbol{k}_3=(2,-1)$, $\boldsymbol{k}_4=(1,2)$, $\boldsymbol{k}_5=(0,1)$, $\boldsymbol{k}_6=(1,0)$, $\boldsymbol{k}_7=(1,-2)$, $\boldsymbol{k}_8=(3,1)$, $\boldsymbol{k}_9=(2,2)$ and put $\gamma_{\boldsymbol{k}_1}=\gamma_1$, $\gamma_{\boldsymbol{k}_2}=-\mathrm{i}\gamma_2$, $\gamma_{\boldsymbol{k}_3}=\gamma_3$, $\gamma_{\boldsymbol{k}_4}=\mathrm{i}\gamma_4$, $\gamma_{\boldsymbol{k}_5}=\gamma_5$, $\gamma_{\boldsymbol{k}_6}=\mathrm{i}\gamma_6$, $\gamma_{\boldsymbol{k}_7}=\mathrm{i}\gamma_7$, $\gamma_{\boldsymbol{k}_8}=\gamma_8$, $\gamma_{\boldsymbol{k}_9}=\mathrm{i}\gamma_9$.

The equations for γ_j with $j=1,\ldots,9$ become, after rescaling by a factor $(10)^{1/2}$, letting $\nu=1$ (this is equivalent to a change of time and length units) and assuming the forcing only on the mode $\boldsymbol{k}_3$:

$$\begin{aligned}
\frac{d\gamma_1}{dt} &= -2\gamma_1+4\gamma_2\gamma_3+4\gamma_4\gamma_5 \\
\frac{d\gamma_2}{dt} &= -9\gamma_2+3\gamma_1\gamma_3+9\gamma_5\gamma_8+3\gamma_7\gamma_9 \\
\frac{d\gamma_3}{dt} &= -5\gamma_3-7\gamma_1\gamma_2+\frac{9}{\sqrt{5}}\gamma_1\gamma_7-5\gamma_4\gamma_8+Re \\
\frac{d\gamma_4}{dt} &= -5\gamma_4-\gamma_1\gamma_5+5\gamma_3\gamma_8+7\gamma_6\gamma_9 \\
\frac{d\gamma_5}{dt} &= -\gamma_5-3\gamma_1\gamma_4+\sqrt{5}\gamma_1\gamma_6-\gamma_2\gamma_8 \\
\frac{d\gamma_6}{dt} &= -\gamma_6-\sqrt{5}\gamma_1\gamma_5-3\gamma_4\gamma_9 \\
\frac{d\gamma_7}{dt} &= -5\gamma_7-\frac{9}{\sqrt{5}}\gamma_1\gamma_3+\gamma_2\gamma_9 \\
\frac{d\gamma_8}{dt} &= -10\gamma_8-8\gamma_2\gamma_5 \\
\frac{d\gamma_9}{dt} &= -8\gamma_9-4\gamma_2\gamma_7-4\gamma_4\gamma_6.
\end{aligned} \tag{A1.3}$$

Now $Re \propto f_3$ is the Reynolds number, the only control parameter for (A1.3). There exist systematic studies on equations (A1.3), considering 5, 6, 7, 8 and 9 modes; see [9] for a review. In these works the following behaviours are observed at increasing Reynolds number Re:

(i) stable fixed points;

(ii) Hopf bifurcations to periodic cyclic orbits;

(iii) periodic/aperiodic/chaotic orbits.

For large Reynolds number (i.e. the case (iii)) the behaviours are strongly dependent on truncation. In our paper we have considered only the truncated equations with five modes ($j=1,\ldots,5$) and seven modes ($j=1,\ldots,7$). It is worth noting that the five-modes model is regular at very large Re, while the seven-modes one is chaotic in the same limit.

Appendix 2

We want here to discuss an old and often debated problem of the physics of fluids which is related to the results obtained in this paper. We are actually interested in the growth of the distance $l(t)$ of two particles which are initially close in the fluid. Batchelor [12] was one of the first to argue, by non-rigorous but reasonable arguments, that in the limit of large times and $l(t=0)\to 0$:

$$l(t) \propto l(0)\, e^{\alpha t} \qquad \text{with } \alpha > 0. \tag{A2.1}$$

This conclusion has been disputed by Cocke [13] who claimed that (A2.1) does not hold. His arguments are quoted as correct even in one of the main textbooks of fluid dynamics [14]. On the contrary, we think that (A2.1) does not disagree with any general principle and clearly follows by Lagrangian chaos. Indeed α is just the LE λ_L defined in (5).

Cocke's conclusion is due to an exchange of limits. As he proved that under general assumptions, in a turbulent fluid at large time:

$$l(t) \propto t^{1/2} \tag{A2.2}$$

and therefore $t^{-1} \ln l(t) \to 0$, he concluded that (A2.1) could not hold. However, there is no disagreement between (A2.1) and (A2.2) because (A2.2) is derived at large times with fixed $l(0)$, i.e. $t \gg |\ln l(0)|$. On the other hand, the limit $l(0) \to 0$ is taken before the limit $t \to \infty$ in (A2.1) and we think that this is the origin of the misunderstandings in the matter.

Let us finally remark that the two different behviours (A2.1) at 'small' times and (A2.2) at 'large' times can be easily observed in numerical experiments [22].

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